Mathematical modelling of wind turbines

A.Ibrahim (UKZN)
K.Kanapi, N.Whittaker, A.Theophilopoulos
D.P. Mason (WITS)

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Outline

The Betz Limit

The full wake

Turbulent far wake

- $\nu \neq 0$
- $\nu \to 0$
- $\nu = 0$
- How should we space wind turbines?

Stress analysis of blades

$$C_P = \frac{P_R}{P_W} = 59.259\%$$

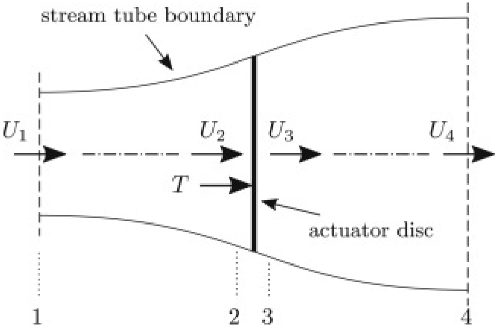


Figure: Simplified model used in the Betz limit.

We found using a similarity solution approach that,

$$w = F(\xi), \qquad v_r = \frac{1}{\sqrt{z}} G(\xi), \qquad \xi = \frac{r}{\sqrt{z}}.$$

Conserved quantities

$$D = z \int_0^\infty \xi F(\xi) d\xi, \qquad J = z \int_0^\infty \xi F^2(\xi) d\xi.$$

$$v_z = \bar{v}_z + v'_z, \qquad\qquad v_r = \bar{v}_r + v'_r.$$

$$v_z = U_0 - w$$

Kinematic Eddy viscosity

$$\nu_T = \ell^2(z)\left|\frac{\partial v_z}{\partial r}\right|. \tag{1}$$

Far wake approximation

1. We drop products of $\bar{w}$ and/or $\bar{v}_r$.
2. Retain the ν_T term.

$$\bar{w} = z^{-1}F(\xi), \qquad \bar{v}_r = z^{-\frac{3}{2}}G(\xi), \qquad \xi = \frac{r}{\sqrt{z}}.$$

The wake boundary and Prandtl's mixing length:

$$b(z) = \beta z^{\frac{1}{2}}, \qquad \ell(z) = \ell_0 z^{\frac{3}{4}}.$$

$$D = U_0 \int_0^\beta \xi F(\xi) d\xi, \tag{2}$$

$$G(\xi) = -\frac{1}{2}\xi F(\xi), \tag{3}$$

$$F'(\xi) = \frac{\nu - \sqrt{\nu^2 + 2\ell_0^2 U_0 \xi F}}{2\ell_0^2}. \tag{4}$$

$$F(\xi) = \frac{U_0}{18\ell_0^2}\left(\beta^{\frac{3}{2}} - \xi^{\frac{3}{2}}\right)^2, \qquad 0 \le \xi \le \beta,$$

where,

$$\beta = \left(\frac{140\ell_0^2 D}{U_0^2}\right)^{\frac{1}{5}}.$$

$$\bar{w} = z^{-\frac{2}{3}} F(\xi), \qquad \bar{v}_r = z^{-\frac{4}{3}} G(\xi), \qquad \xi = \frac{r}{z^{\frac{1}{3}}}.$$

The wake boundary and Prandtl's mixing length:

$$b(z) = \beta^* z^{\frac{1}{3}}, \qquad \ell(z) = \ell_0^* z^{\frac{1}{3}}.$$

$$F(\xi) = \frac{U_0}{27\ell_0^{*2}} \left(\beta^{*\frac{3}{2}} - \xi^{\frac{3}{2}}\right)^2, \qquad 0 \leq \xi \leq \beta^*,$$

where,

$$\beta^* = \left(\frac{210\ell_0^{*2} D}{U_0^2}\right)^{\frac{1}{5}}.$$

Plots

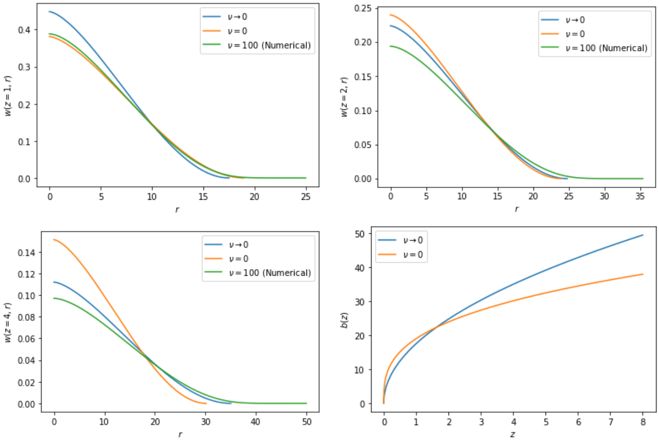
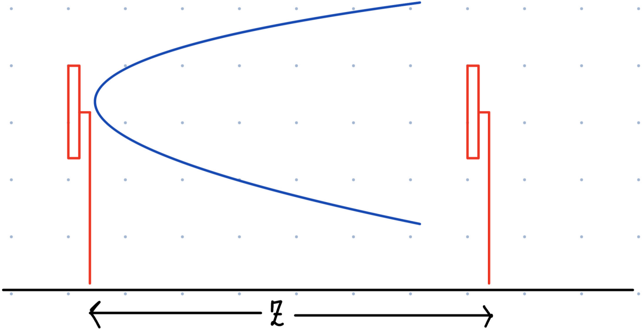


Figure: Numerical comparisons

How should we space wind turbines?



$$\frac{\bar{w}_z^{max}}{U_0} \leq 1\%$$

How should we space wind turbines?

$\nu \neq 0:$

$$z \geq \frac{100F(0)}{U_0},$$

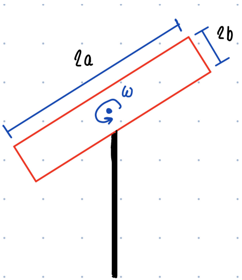
$\nu \to 0:$

$$z \geq \frac{50\beta^3}{9\ell_0^2},$$

$\nu = 0:$

$$z \geq \left(\frac{100\beta^{*3}}{27\ell_0^{*2}}\right)^{\frac{3}{2}}.$$

Stress analysis of blades



$$F_{net} = F + F^*,$$

where,

$$F^* = -\left(\frac{d\underline{\omega}}{dt}\right)_R \times \underline{r} - \underline{\omega} \times (\underline{\omega} \times \underline{r}).$$

Boundary conditions:

1. Blade edge is traction free.
2. Tip of blades have no normal/shear force or moment.

Let $\bar{x} = \frac{x}{a}$ and $\bar{y} = \frac{y}{b}$.

Stresses:

$$\tau_{xx} = \frac{1}{2}\rho\omega^2 a^2 \left[1 - \bar{x}^2 + 2\sigma\left(\frac{b}{a}\right)^2\left(\frac{1}{3} - \bar{y}^2\right)\right],$$

$$\tau_{xy} = 0,$$

$$\tau_{yy} = \frac{1}{2}\rho\omega^2 b^2(1 - \bar{y}^2).$$

Maximum normal stress

$$\tau^{max} = \frac{1}{2}\rho\omega^2 a^2 \left[1 + \frac{2\sigma}{3}\left(\frac{b}{a}\right)^2\right]$$

This occurs at $(\bar{x}, \bar{y}) = (0, 0)$.

Maximum shearing stress

$$S_{max} = \frac{1}{4}\rho\omega^2 a^2 \left[1 - \frac{4\sigma}{3}\left(\frac{b}{a}\right)^2\right]$$

This occurs at $(\bar{x}, \bar{y}) = (0, \pm 1)$.

Maximum displacement

The maximum displacement occurs at $(\bar{x}, \bar{y}) = (\bar{x}_0, 0)$,

$$\bar{x}_0 = \pm\sqrt{1 - \frac{\sigma}{3}\left(\frac{b}{a}\right)^2},$$

with magnitude,

$$u_x^{max} = \pm\frac{\rho\omega^2 a^3}{3E}\left[1 - \frac{\sigma}{3}\left(\frac{b}{a}\right)^2\right]^{\frac{3}{2}}.$$

Note:

$$\left(\frac{b}{a}\right)^2 \approx 10^{-3}$$

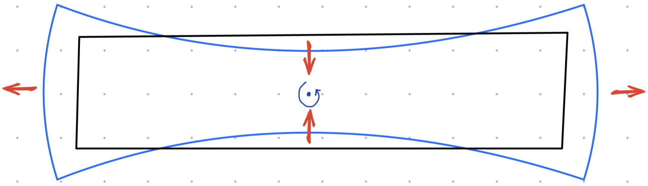


Figure: Deformed beam

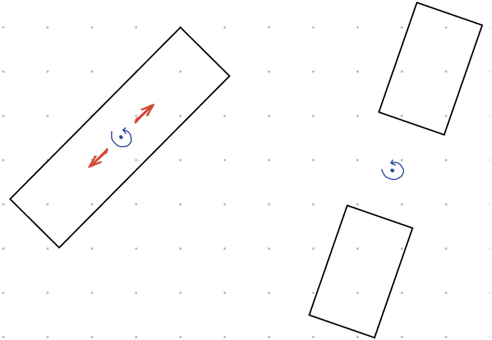


Figure: Normal stress fracture.

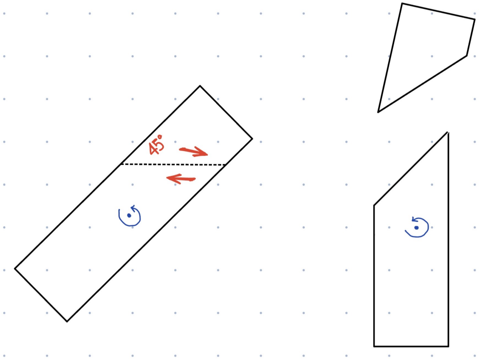


Figure: Shear stress fracture.

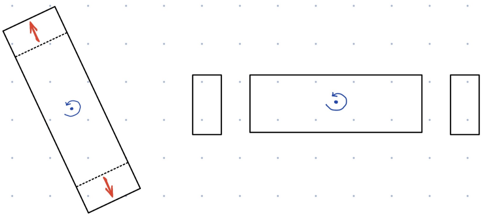


Figure: Extension stress fracture.